

Pseudopotentials for Multi-particle Interactions in the Quantum Hall Regime

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In fractional quantum Hall physics, the Hilbert space is projected to a single Landau level and the entire Hamiltonian consists of just the projected inter-electron interaction. Haldane's pseudopotential formalism has been an extremely useful tool both for understanding these interactions, and for understanding the quantum Hall states that result. In the current paper we consider the analogue of this pseudopotential construction that results from general M -body interactions rather than the usual (Coulomb) two-body interaction.

I. INTRODUCTION

In high magnetic fields, the electronic states of two dimensional systems break up into degenerate bands known as Landau levels¹. In high enough fields, i.e., in the fractional quantum Hall regime, one assumes that the spacing between these levels is large enough such that all of the degrees of freedom are restricted to a single Landau level. In this case, the entire Hamiltonian of the system is simply the projected inter-particle interaction. In an extremely insightful early paper by Haldane^{1,2}, it was pointed out that any translationally and rotationally invariant two-body interaction, projected to a single Landau level, could be completely described by a set of "pseudopotential coefficients" (V_L with $L \geq 0$) describing the energy cost to have a pair of particles in a state of relative angular momentum L (for electrons L must be odd whereas if we were considering quantum Hall effect of bosons³ L would have to be even). This formalism turned out to be useful not just for describing the details of the interaction, but also for describing the resulting fractional quantum Hall states. For example, the Laughlin $\nu = 1/3$ state is precisely described as the unique highest density zero energy state of a potential which has V_1 positive and $V_L = 0$ for all $L > 1$. Furthermore, using the pseudopotential formalism, an interacting system in a partially filled higher Landau level can be mapped to an equivalent problem in the lowest Landau level with a modified interaction¹.

In recent years, the study of the fractional quantum Hall effect has begun to consider interactions beyond just Coulomb two-body interaction. Indeed, a particularly interesting class of quantum Hall states, including the Moore-Read state⁴ and the Read-Rezayi⁵ states are exact highest density zero energy states of M -body interactions with $M > 2$. More recent work^{6,7,8} has considered even more complicated M -body interactions. These more complicated many particle interactions may not just be theoretical tools. For example, it has been shown⁹ how multi-particle interactions may be engineered in trapped

atom systems. More generally, multi-particle interactions could easily result from integrating out virtual excitations of any sort. Furthermore, while the above mentioned quantum Hall states are exact ground states for very specific many-particle interactions, they may also be extremely good trial states for more realistic interactions¹⁰. More importantly, in many cases, these exact ground states provide a particularly simple (and easily studied) representative of an entire phase of matter.

Despite the increasing interest in these multi-particle interactions, generalizations of Haldane's formalism have not, to our knowledge, been systematically constructed. (Note, however, that some progress in this direction has been given in Ref. 11 in a somewhat different language). Performing this construction explicitly is the main objective of the current paper.

The outline of this paper is as follows. In section II we start with a brief review of Haldane's pseudopotential construction for simple two-body interactions. In Section III we consider how this construction should be generalized to M -body interactions. We find that the construction is more complicated than the two-body case because the simple pseudopotential coefficients need to be generalized to pseudopotential matrices. Section IV is devoted to calculating the dimension of the resulting matrices, and specifying a convenient basis in which to express these interaction matrices. In section V, we briefly discuss how these results might be used and why they are interesting. For clarity, some simple examples of using these results for three-body interactions are given in detail in the final appendix.

II. REVIEW OF HALDANE'S CONSTRUCTION

Before launching into the general construction for M -body interactions, it is useful to review how Haldane's formalism works for two-body interactions^{1,2} (albeit in a slightly different language than that of Haldane's original

construction). Projected to a single Landau level, the Hamiltonian consists of an interaction term only

$$H = \sum_{i < j} V(\mathbf{r}_i - \mathbf{r}_j) \quad (1)$$

It will be crucial that the interaction term is both rotationally and translationally invariant.

Since the Hamiltonian operates on two electrons at a time, we will consider two electron wavefunctions of the form $\Psi(\mathbf{r}_1, \mathbf{r}_2)$. This is sufficient to analyze the effect of the Hamiltonian in general since any many electron wavefunction can be decomposed into a sum of a two particle wavefunction times a wavefunction of everything else. For example, to find the effect of the term $V(\mathbf{r}_1 - \mathbf{r}_2)$ on a general multiparticle wavefunction Ψ , we can always decompose

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_a \Psi_a(\mathbf{r}_1, \mathbf{r}_2) \tilde{\Psi}_a(\mathbf{r}_3, \dots, \mathbf{r}_N) \quad (2)$$

and clearly the $V(\mathbf{r}_1 - \mathbf{r}_2)$ term of the Hamiltonian only operates on the Ψ_a term and not the $\tilde{\Psi}_a$ term. Thus, we need only focus on two electron wavefunctions $\Psi(\mathbf{r}_i, \mathbf{r}_j)$.

It is then useful to further decompose the two electron wavefunction into center of mass and relative degrees of freedom

$$\Psi(\mathbf{r}_i, \mathbf{r}_j) = \sum_{b,c} A_{b,c} \Psi_b^{CM} \left(\frac{\mathbf{r}_i + \mathbf{r}_j}{2} \right) \Psi_c^{rel}(\mathbf{r}_i - \mathbf{r}_j) \quad (3)$$

where Ψ_b^{CM} and Ψ_c^{rel} form a basis for the center of mass and relative wavefunctions respectively. We note that the Hamiltonian, which is translationally invariant, acts only on the relative coordinate.

In the lowest Landau level, using the usual analytic notation for wavefunctions¹ where single particle orbitals are given by $z^m \exp(-|z|^2/4)$, and the magnetic length is taken to be unity, we can thus rewrite any product of the wavefunctions Ψ_b^{CM} and Ψ_c^{rel} as

$$\psi_b^{CM} \left(\frac{z_i + z_j}{2} \right) \psi_c^{rel}(z_i - z_j) e^{-\frac{1}{4}(|z_1|^2 + |z_2|^2)} \quad (4)$$

where we have separated out the Gaussian factors. Both ψ^{CM} and ψ^{rel} must be analytic polynomials, and sometimes we will notate ψ^{rel} as simply ψ . Throughout this paper we will be discussing the planar (disk) geometry. In Appendix A other geometries (sphere and torus) are briefly discussed.

A convenient complete basis to describe the possible relative wavefunctions ψ^{rel} is now given by

$$|L; i, j\rangle = C_L (z_i - z_j)^L \quad (5)$$

where L is the relative angular momentum between particles i and j and C_L is an appropriate normalization constant. Note that if the particles are fermions then L

must be odd, and if they are bosons, then L must be even. We can now insert this complete set into the Hamiltonian

$$H = \sum_{i < j} \sum_{L, L'} |L; i, j\rangle \langle L; i, j| V(\mathbf{r}_i - \mathbf{r}_j) |L'; i, j\rangle \langle L'; i, j| \quad (6)$$

Note that since the interaction is translationally invariant, we need only insert a complete set for the relative degrees of freedom (I.e., the interaction is diagonal and trivial in the center of mass degree of freedom).

The rotational invariance of the interaction now makes the matrix element diagonal in L so we obtain the Haldane Hamiltonian^{1,2}

$$H = \sum_{i < j} \sum_L V_{L,2} P_{ij}^L \quad (7)$$

where

$$P_{ij}^L = |L; i, j\rangle \langle L; i, j| \quad (8)$$

is a projection operator that projects particles i and j to a state of relative angular momentum L (within the lowest Landau level). In Eq. 7,

$$V_{L,2} = \langle L; i, j| V(\mathbf{r}_i - \mathbf{r}_j) |L; i, j\rangle \quad (9)$$

$$= \langle L| V |L\rangle \quad (10)$$

is known as the pseudopotential coefficient, and we have added the subscript 2 here to indicate that this is a two-body interaction. In Eq. 10 we have written this matrix element in a convenient shorthand, since by symmetry between particles, $V_{L,2}$ is independent of which i and j is chosen.

One of the great advantages of the pseudopotential formalism is how easily it generalizes to higher Landau levels^{1,12} as well as to even more complicated situations¹³. The simple generalization stems from the fact that in any Landau level, there is a one-to-one mapping to another system completely in the lowest Landau level with a modified inter-electron interaction^{1,12,13}. To see how this mapping works, we consider a system in the first excited Landau level (1LL). Here, one can write any single electron state as a Landau level raising operator a^\dagger applied to a lowest Landau level state. In this way, an orthogonal set of relative angular momentum states analogous to Eq. 5 may be written in the 1LL

$$|L; i, j; 1LL\rangle = a_i^\dagger a_j^\dagger |L; i, j\rangle \quad (11)$$

where¹²

$$a_i^\dagger = \sqrt{2}(-\partial_{z_i} + z_i^*/4) \quad (12)$$

where the derivative acts on the Gaussian factors as well as on the polynomial part of the wavefunction (and * means complex conjugation). Using this orthogonal set for states in the 1LL, and following the same argument, we obtain exactly the same Haldane Hamiltonian (Eq. 7) but with modified pseudopotential coefficients

$$V_{L,2} = \langle L; i, j| a_i a_j V(\mathbf{r}_i - \mathbf{r}_j) a_i^\dagger a_j^\dagger |L; i, j\rangle. \quad (13)$$

Further discussion of Landau level raising (for the case of three-body interactions) is given in example 4 of appendix E.

III. M -BODY INTERACTIONS

We now generalize Haldane's argument to M -body interactions. We thus consider a general M -body interaction Hamiltonian

$$H = \sum_{i_1 < i_2 < \dots < i_M} V(\mathbf{r}_1, \dots, \mathbf{r}_M) \quad (14)$$

We note that, although not a requirement, we will typically want to think about M -body interactions that are not reducible to sums of interactions of a smaller number of particles. This issue is discussed in more depth in appendix B.

Again crucial to our construction is the assumption that V is both translationally and rotationally invariant. We then consider a general M -body wavefunction $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_M)$, which we rewrite in terms of relative and center of mass coordinates

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_M) = \sum_{b,c} A_{b,c} \quad (15)$$

$$\Psi_b^{CM} \left(\frac{\mathbf{r}_1 + \mathbf{r}_2 + \dots + \mathbf{r}_M}{M} \right) \Psi_c^{rel}(\mathbf{r}_1, \dots, \mathbf{r}_M)$$

The statement that Ψ^{rel} is a wavefunction for relative motion means it is translationally invariant, or

$$\begin{aligned} \Psi^{rel}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_M) = \\ \Psi^{rel}(\mathbf{r}_1 + \mathbf{x}, \mathbf{r}_2 + \mathbf{x}, \dots, \mathbf{r}_M + \mathbf{x}) \end{aligned} \quad (16)$$

for arbitrary \mathbf{x} .

Once again, we can assume we are in the lowest Landau level so we can work with analytic forms of the wavefunction, so we write the product $\Psi^{CM} \Psi^{rel}$ as

$$\psi^{CM} \left(\frac{1}{M} \sum_{i=1}^M z_i \right) \psi(z_1, \dots, z_M) e^{-\frac{1}{4} \sum_{i=1}^M |z_i|^2} \quad (17)$$

where ψ is then the relative wavefunction, which must be a translationally invariant polynomial (and must be antisymmetric for fermions and symmetric for bosons). As in the two-body case, we can categorize these relative wavefunctions in terms of an angular momentum quantum number. We define the “relative angular momentum” operator on the relative wavefunction ψ to be the total degree of the complex polynomial ψ . If the polynomial ψ is homogeneous of degree L , it is a relative angular momentum eigenstate with eigenvalue L . Once again we note that our construction here pertains to the planar (disk) geometry. The sphere and torus are discussed briefly in appendix A.

As in the two-body case, we are now seeking a complete basis in which to write our relative wavefunctions. We

can certainly write any relative wavefunction ψ as a sum of components with all possible values of $L \geq 0$. I.e.,

$$\psi(z_1, \dots, z_M) = \sum_L \psi_L(z_1, \dots, z_M) \quad (18)$$

where ψ_L has relative angular momentum L , i.e, is homogeneous of degree L . Unfortunately, as compared to the two-body case, it is no longer true that for a given L there is only a single possible wavefunction. Indeed, as we will see in great detail in section IV below, there may be many M -body wavefunctions with the same relative angular momenta. Thus, within the space of states with relative angular momentum L , we arbitrarily define an orthonormal basis which we write as $|L, q\rangle$. The dimension d of this space (and hence the number of different values of q) depends on L , M and whether we are considering bosons or fermions. In section IV we will determine these dependencies.

As in the two-body case, since the interaction is independent of the center of mass of the M particles, we can then insert this complete set of relative wavefunctions into the Hamiltonian to obtain

$$H = \sum_{i_1 < \dots < i_M} \sum_{L, q, q'} |L, q; i_1, \dots, i_M\rangle V_{L, M}^{q, q'} \langle L, q'; i_1, \dots, i_M| \quad (19)$$

which is a sum over all possible M -clusters. Here the matrix element is a Hermitian matrix

$$\begin{aligned} V_{L, M}^{q, q'} &= \langle L, q; i_1, \dots, i_M | V(\mathbf{r}_1, \dots, \mathbf{r}_M) | L, q'; i_1, \dots, i_M \rangle \\ &= \langle L, q | V | L, q' \rangle \end{aligned} \quad (20)$$

where in the second line we have not written the dependence on i_1, \dots, i_M since by symmetry again it does not matter which particles i_1, \dots, i_M we choose to use to calculate $V_{L, M}^{q, q'}$. Again, we use the subscript M here to indicate that we are considering an M -body interaction. Note that this matrix element is necessarily diagonal in L by rotational invariance¹⁴ of the interaction V , but is not generally diagonal in q . For a given L , if there are d different basis vectors $|L, q\rangle$, one must then specify the full d by d Hermitian matrix $V_{L, M}^{q, q'}$.

At this point, it is useful to point out that we need not work with an orthonormal basis $|L, q\rangle$. Even given a nonorthonormal (but linearly independent) basis $|L, r\rangle$ we can construct an orthonormal q -basis as

$$|L, q\rangle = \sum_r [R^{-1/2}]_{qr} |L, r\rangle \quad (21)$$

where the hermitian matrix R is given by

$$R_{r, r'} = \langle L, r | L, r' \rangle \quad (22)$$

Thus, we can freely translate our Hamiltonian into an arbitrary $|L, r\rangle$ basis as

$$H = \sum_{i_1 < \dots < i_M} \sum_{L, r, r'} |L, r; i_1, \dots, i_M\rangle V_{L, M}^{r, r'} \langle L, r'; i_1, \dots, i_M| \quad (23)$$

where now

$$V_{L,M}^{r,r'} = \sum_{r_1, r'_1} [R^{-1}]_{r'_1, r'} [R^{-1}]_{r, r_1} \langle L, r'_1 | V | L, r_1 \rangle \quad (24)$$

The form of Eq. 23 is generally the analogue of the Haldane Hamiltonian for M -body interactions, and is applicable for any complete linearly independent basis $|L, r\rangle$ (although it may in fact be convenient to choose to work with an orthonormal basis).

While this result seems relatively straightforward, it is somewhat nontrivial to find a basis $|L, r\rangle$ (or even to compute its dimension). This is precisely the task we will undertake in the next section.

IV. SPACE OF FIXED ANGULAR MOMENTUM RELATIVE WAVEFUNCTIONS

We would like to examine the space of M -particle translationally invariant wavefunctions ψ with relative angular momentum L . In other words, we want to study analytic functions in M variables which are homogeneous of degree L and are antisymmetric for the case of fermions, or symmetric for the case of bosons.

Let us start by quoting a result to be proven below which applies directly to the bosonic case: The dimension of the space of of translationally invariant symmetric polynomials in M variables which are homogeneous of degree L is given by

$$d_{sym}(L, M) = \begin{array}{l} \text{number of partitions of the integer } L \\ \text{into pieces no larger than } M \text{ which} \\ \text{do not include the integer 1} \end{array} \quad (25)$$

For example, we have $d_{sym}(7, 4) = 2$ since there are only two partitions of 7 into pieces larger than 1 and no larger than 4 (These partitions are obviously $7=3+4$ and $7=3+2+2$). A table of $d_{sym}(L, M)$ is given explicitly for small values of M and L in table I. In Appendix C an explicit analytic form is given for this quantity. The proof of Eq. 25 will be performed below by constructing an explicit basis for these symmetric homogeneous translationally invariant polynomials.

To handle the fermion case, it is useful to use the result that any antisymmetric polynomial can be written as a Vandermonde determinant times a symmetric polynomial¹⁵

$$\psi_{antisym}(z_1 \dots z_M) = \left[\prod_{i < j} (z_i - z_j) \right] \psi_{sym}(z_1 \dots z_M) \quad (26)$$

Thus, there is a one to one mapping from the space of homogeneous symmetric polynomials of degree $L = p$ in M variables to the space of antisymmetric polynomials of degree $L = p + (M(M-1))/2$ in M variables. It is therefore sufficient to consider only the symmetric case, and

$L =$	0	1	2	3	4	5	6	7	8	9
$M = 2$	1	0	1	0	1	0	1	0	1	0
$M = 3$	1	0	1	1	1	1	2	1	2	2
$M = 4$	1	0	1	1	2	1	3	2	4	3
$M = 5$	1	0	1	1	2	2	3	3	5	5
$M = 6$	1	0	1	1	2	2	4	3	6	6
$M = 7$	1	0	1	1	2	2	4	4	6	7
$M = 8$	1	0	1	1	2	2	4	4	7	7
$M = 9$	1	0	1	1	2	2	4	4	7	8

TABLE I: Table of $d_{sym}(L, M)$, the dimension of the space of homogeneous symmetric translationally invariant polynomials of degree L in M variables. The formula for these entries is Eq. C1 or equivalently Eq. 25. The dimension of the space of homogeneous antisymmetric translationally invariant polynomials of degree L in M variables is given in terms of this table by $d_{antisym}(L, M) = d_{sym}(L - M(M-1)/2, M)$ (see Eq. 27). Note that for an M -body interaction, at angular momentum L , there is a d by d Hermitian pseudopotential matrix ($d = d_{sym}(L, M)$ for bosons and $d_{antisym}(L, M)$ for fermions).

the dimension of antisymmetric homogeneous functions of degree L in M variables is given by

$$d_{antisym}(L, M) = d_{sym}(L - M(M-1)/2, M). \quad (27)$$

We now proceed to construct an explicit basis for the homogeneous translationally invariant symmetric polynomials of degree L in M variables. To do this, we begin with the well known elementary symmetric polynomials¹⁵

$$e_{m,M} = e_{m,M}(z_1, \dots, z_M) = \sum_{0 < i_1 < i_2 < \dots < i_m \leq M} z_{i_1} \dots z_{i_m} \quad (28)$$

for $m \leq M$ and $e_{m,M}$ is defined to be zero for $m > M$. The products of these elementary polynomials form a linearly independent basis for all symmetric polynomials in M variables. (I.e., the elementary symmetric polynomials generate the mathematical *ring* of symmetric polynomials.) We can write this basis explicitly as

$$v_{\lambda, M} = \prod_{j=1}^M [e_{j,M}(z_1, \dots, z_M)]^{\lambda_j} \quad (29)$$

where λ is a vector of nonnegative integers λ_j with $j = 1, \dots, M$. The degree of the basis vector $v_{\lambda, M}$ is $L = \sum_{j=1}^M j\lambda_j$. Thus, we can think of each different λ as representing a partition of the integer L into pieces with each piece no larger than M . For example, the vector $\lambda = (2, 1, 2, 3)$ represents the partition $(1, 1, 2, 3, 3, 4, 4, 4)$ which corresponds to a homogeneous symmetric polynomial of degree $L = 22$. Thus the dimension of the space of symmetric homogeneous polynomials of degree L in M variables is given by the number of partitions of the integer L into pieces no larger than M (denoted $P_{L, M}$ in appendix C).

The subset of polynomials we are interested in are, in addition, translationally invariant. Any such polynomial can be written as a function only of the relative coordinates

$$\tilde{z}_i = z_i - \frac{1}{M} \sum_{j=1}^M z_j \quad (30)$$

We thus consider as our basis polynomials generated by products of $e_{m,M}(\tilde{z}_1, \dots, \tilde{z}_M)$. Note however that since $e_{1,M} = e_{1,M}(z_1, \dots, z_M) = \sum_{j=1}^M z_j$ we have $e_{1,M}(\tilde{z}_1, \dots, \tilde{z}_M) = 0$. Thus, our new basis is of the form

$$w_{\lambda,M} = \prod_{j=2}^M [e_{j,M}(\tilde{z}_1, \dots, \tilde{z}_M)]^{\lambda_j} \quad (31)$$

which for a given λ is of degree $L = \sum_{j=2}^M j\lambda_j$. Analogous to above λ now represents a partition of L into pieces no larger than M but now none of the pieces are allowed to be 1 (i.e., we can take $\lambda_1 = 0$). The size of this basis is therefore precisely given by the rule quoted above in Eq. 25.

The $w_{\lambda,M}$ are constructed to span the space of translationally invariant symmetric polynomials. However, we must also prove that the all of the $w_{\lambda,M}$ basis elements are linearly independent (This is nontrivial since there is a linear relation between all of the \tilde{z} 's). This proof is more involved and is relegated to appendix D.

V. DISCUSSION

As in the original Haldane² construction of pseudopotentials for two-body interactions, one of the main uses of our result is to describe any interaction in a maximally simplified parameterized form. However, as with the Haldane case, examining the form of the pseudopotential coefficients (and in particular, examining table I) can have some interesting ramifications in terms of what type of simple quantum Hall states might exist. To see this connection, let us revisit the Laughlin states for a moment. As pointed out by Haldane's original work², the Laughlin state can be thought of as the exact highest density zero energy state of an interaction with only certain nonzero pseudopotentials. For simplicity here, let us continue to think about quantum Hall effect of bosons (i.e., symmetric quantum Hall wavefunctions). If we consider two-body potentials, examining table I we see that $d_{sym}(L, 2) = 1$ for even L and is zero for odd. Thus there is a single pseudopotential coefficient for even values of L , and no coefficient for L odd. I.e., to specify a two-body potential we specify the coefficients $V_{0,2}, V_{2,2}, V_{4,2}, \dots$ (The traditional notation is V_0, V_2, V_4, \dots). Recall the additional subscript 2 here refers to the fact that we are considering a two-body interaction). Haldane's interesting realization² is that the Laughlin bosonic $\nu = 1/2$ state is the exact highest density zero energy state of a potential that has $V_{0,2} > 0$

and all other V 's equal to zero. Similarly, if $V_{0,2} > 0$ and $V_{2,2} > 0$, and all other coefficients are zero, the highest density zero energy state is the Laughlin bosonic $\nu = 1/4$ state. These results are, of course, well known. We emphasize that to obtain these Laughlin states we are simply forbidding any two particles from having a relative angular momentum below some cutoff.

Let us now consider three-body interactions for bosons. Examining our table, we see that $d_{sym}(L, 3) = 1$ for $L = 0, 2, 3, 4, 5, 7$. Thus, for each of these values of L , there is a single pseudopotential coefficient (for $L = 1$ there is no state, since $d_{sym}(1, 3) = 0$ and therefore no coefficient). In analogy with Haldane's original paper, one might naturally ask questions like what is the highest density zero energy state of a system with $V_{0,3} > 0$ and all other pseudopotential coefficients equal to zero. In other words, what happens if we forbid any three particles from having zero relative angular momentum. (Note that this $V_{0,3}$ pseudopotential coefficient also corresponds to a three-body point-contact interaction, as demonstrated in appendix E). It is well known that the highest density zero energy state of such an interaction is the bosonic Moore-Read Pfaffian^{4,16} at filling fraction $\nu = 1$.

We could similarly ask what happens if we make $V_{0,3} > 0$ and $V_{2,3} > 0$, and all other coefficients zero. Thus we are asking what happens if we forbid three particles from having relative angular momentum 0 or 2 (there is no state where three particles have relative angular momentum 1). The highest density zero energy state of this interaction, known as the "Gaffnian", occurs at filling fraction $\nu = 2/3$ for bosons and has been studied in depth in Ref. 7. (In appendix E a form of this interaction is given as derivatives of delta functions). Similarly, if $V_{0,3}, V_{2,3}$, and $V_{3,3}$ are positive and all other coefficients are zero the highest energy zero energy wavefunction of such an interaction is known as the Haffnian and has been studied in depth in Ref. 6. In each case here, we are simply forbidding three particles from having relative angular momentum below a certain cutoff and we can consider successively increasing this cutoff.

One can also generalize this construction to other values of M . For example, the Read-Rezayi state⁵ at filling fraction $\nu = (M-1)/2$ for bosons, is the highest density zero energy state of a Hamiltonian that consists only of a positive $V_{0,M}$ coefficient and all other coefficients are zero. The generalization of this construction — forbidding clusters of M particles to have angular momentum below some cutoff L — is studied by the current authors in Ref. 8.

When $d_{sym}(L, M) > 1$ there is a d_{sym} dimensional pseudopotential matrix $V_{L,M}^{q,q'}$. For simplicity, let us assume that we are working in a basis where the wavefunctions $|L, q\rangle$ are orthogonal states (recall that each $|L, q\rangle$ is an M -particle relative wavefunction). If we were to choose the $V_{L,M}^{q,q'}$ matrix to be positive definite (all eigenvalues strictly positive), it would forbid (give positive energy

to) any cluster of M particles with relative angular momentum L . This is the situation that is considered in Ref. 8.

However, we could also consider more interesting situations. If we choose $V_{L,M}^{q,q'}$ to have some eigenvectors with positive eigenvalue and some eigenvectors with zero eigenvalue, we allow only certain clusters of M particles with angular momentum L while disallowing (giving positive energy to) others. To be more specific, any wavefunction describing a cluster of M -particles that lies completely in the space of zero eigenvalue eigenvectors (the “null space”) is a zero energy cluster, whereas any cluster that has components outside of this null-space has positive energy. (An explicit example of this is given in Appendix E).

It is particularly interesting to consider the situation where the matrix $V_{L,M}^{q,q'}$ has a single eigenvector $|\phi\rangle$ with zero eigenvalue (and all other eigenvalues are positive). What this means is that in the $d_{sym}(L, M)$ dimensional space of translationally invariant homogeneous polynomials of degree L in M -variables, there is only a single (translationally invariant) polynomial $\phi(z_1, \dots, z_M)$ that is a zero energy M -particle wavefunction, and all other polynomials are positive energy wavefunctions. We may then ask whether a quantum Hall wavefunction exists with the property that all clusters of M particles are in the relative wavefunction ϕ . If such a wavefunction does exist, it would certainly be a zero energy state of our chosen Hamiltonian. In some cases it may also be the highest density zero energy state, i.e., the ground state of the Hamiltonian. In future work, we will show a number of examples where this is indeed the case.

Such a general construction could be quite significant, as it allows us to construct a different quantum Hall wavefunction for each (translationally invariant) cluster function ϕ that we might define. In cases where the dimension $d_{sym}(L, M) = 1$, this is no freedom at all – there is only a single possible cluster function ϕ . However, when $d_{sym} > 1$, we have a continuum of possible choices — and therefore can define a continuum of possible Hamiltonians with a continuum of ground states wavefunctions.

Most interestingly, by judiciously choosing the cluster function ϕ we may be able to find Hamiltonians whose ground state are quantum Hall wavefunctions corresponding to known unitary conformal field theories (For discussion of the connection between quantum hall and conformal field theory see, Refs. 4,5,17). Without proof we believe such constructions are generically possible, and we will show several examples of this in future work. The general belief is that any wavefunction corresponding to a unitary conformal field theory corresponds to a stable phase of matter.

To summarize our work, the main purpose of this paper was to formulate Haldane pseudopotentials for multi-particle interactions. The main technical result of this paper is the enumeration of the possible relative wavefunctions for clusters of M -particles. Since we only consider interactions that are rotationally invariant, we cat-

egorize the possible M -particle translationally invariant cluster wavefunctions by their total angular momentum. For bosons, we find $d_{sym}(L, M)$ basis states whereas for fermions we calculate $d_{antisym}(L, M)$. We discuss how a rotationally invariant M -body interaction can be decomposed into pseudopotential coefficient matrices $V_{L,M}^{q,q'}$. For a given L and M the matrix is d by d (with d being d_{sym} for bosons and $d_{antisym}$ for fermions). We further describe a relatively simple linearly independent (but not orthonormal) basis in which such an interaction matrix could be expressed.

For increased clarity, we would like to show the some simple examples of using this pseudopotential formalism. In particular, we will give details for bosonic systems with three-body potentials. However, since these examples are somewhat lengthy we have put them in Appendix E. We encourage the reader to work through the calculations given there to cement the general ideas of this paper. A brief summary of the appendix is as follows:

- We write down the $w_{\lambda,3}$ basis for three-cluster wavefunctions (as discussed in section IV above)
- Since this basis is not orthonormal, we show how to orthonormalize (as discussed in section III)
- We consider several simple toy model three-body interactions and write down their pseudopotential coefficients in this basis (as discussed in section III)
- We show how to calculate pseudopotential coefficients for higher Landau levels (analogous to that discussed in section II)

The explicit demonstration of these steps in the appendix should make concrete the abstract ideas expressed above.

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APPENDIX A: SPHERE AND TORUS GEOMETRIES

On the spherical geometry², angular momentum is a good quantum number, and any particle in the LLL has angular momentum $S = N_\phi/2$ with N_ϕ the total number of flux quanta through the sphere. The maximum total angular momentum of M particles is MS (if they are bosons). What we called “relative angular momentum” L on the plane corresponds to total angular momentum $MS - L$ on the sphere. For fermions, since the minimum

relative angular momentum is $M(M-1)/2$, the maximum total angular momentum is $MS - M(M-1)/2$. The direction of the “total” angular momentum plays the role of center of mass coordinate on the plane. Interactions that are rotationally invariant, again, will not mix states of differing “relative” angular momentum.

A way to think about the translation from plane to sphere is to realize that any M -particle cluster wavefunction is completely determined by its behavior as all of the particles come close together — the “analyticity” of the wavefunction implies that we know the complete wavefunction over the entire system if we know its behavior for short distances. Thus, if we look at M particles in some relative wavefunction very close together, they do not “see” the curvature of the sphere — locally we can think of the particles as living on a plane, and so our understanding of the possible wavefunctions on the plane can tell us all the possible wavefunctions on the sphere. Once we specify the analytic form of the wavefunction for the particles very close together, this uniquely specifies the wavefunction over the entire sphere, thus generating an appropriate basis for “relative” wavefunctions on the sphere. In other words, for each of our relative wavefunction states on the plane, there is exactly one relative wavefunction on the sphere with the same analytic form at short distances.

In fact, it turns out that there is an extremely simple translation from the plane to sphere. If we point our center of mass coordinate towards the north pole, there is then a simple mapping of the planar wavefunctions (with center of mass at position zero) to the sphere, by stereographic projection⁵. The only slight complication is that the usual gaussian factors on the plane become modified

$$e^{-|z|^2/4} \rightarrow \frac{1}{(1 + |z|^2/4R^2)^{1+S}} \quad (\text{A1})$$

where R is the radius of the sphere.

On the torus, the story is somewhat more complicated, but in principle similar. First of all, on the torus angular momentum is not conserved (for arbitrary unit cell). Nonetheless, we can define a linearly independent basis for relative wavefunctions. As with the above discussion for the sphere when we bring the particles close together, the systems looks locally just like an infinite plane. For each state in our linearly independent basis for the plane there is exactly one state on the torus that has the same short distance limit. Thus, in principle, we can convert our basis on the plane into a basis on the torus.

APPENDIX B: REDUCIBLE INTERACTIONS

Given an M -body interaction $V(\mathbf{r}_1, \dots, \mathbf{r}_M)$ it is frequently convenient to separate out any part of an interaction that is “reducible” to interactions between any lower number $M' < M$ of particles. For example, a three-body

interaction of the form

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V_2(\mathbf{r}_1 - \mathbf{r}_2) + V_2(\mathbf{r}_2 - \mathbf{r}_3) + V_2(\mathbf{r}_1 - \mathbf{r}_3) \quad (\text{B1})$$

is fully reducible to two-body components.

For an arbitrary M -body interaction, it is easy to find the M' -body reducible component (with $M' < M$) by taking appropriate limits where $M - M'$ of the particles are moved away to infinity. For example, for a three-body interaction, the two-body part is given by

$$V_2(\mathbf{r}_1 - \mathbf{r}_2) = \lim_{\mathbf{r}_3 \rightarrow \infty} V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \quad (\text{B2})$$

We then have the irreducible three-body interaction given by

$$V_{irr}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - [V_2(\mathbf{r}_1 - \mathbf{r}_2) + V_2(\mathbf{r}_2 - \mathbf{r}_3) + V_2(\mathbf{r}_1 - \mathbf{r}_3)] \quad (\text{B3})$$

We note that it is possible (although unnecessarily complicated) to describe any two-body interaction as an unreduced three-body interaction. However, typically it is simpler (and clearer) to fully reduce three-body interactions and treat their two-body piece separately. For most realistic potentials, the two-body piece will be very large compared to the reduced three-body piece (which in turn will be large compared to the reduced four-body piece, and so forth). However, by fine-tuning, in certain situations it can be arranged that the pure two-body piece vanishes⁹.

APPENDIX C: ANALYTIC EXPRESSION FOR PARTITIONS

We claim that Eq. 25 can be reexpressed as

$$d_{sym}(L, M) = P_{L,M} - P_{L-1,M} \quad (\text{C1})$$

where $P_{L,M}$ is the number of partitions of the integer L into pieces no larger than M . (Note also that $P_{L,M}$ is equivalently the number of ways to partition the integer L into no more than M pieces).

To see the equivalence between Eq. C1 and Eq. 25 we simply note that given any partition of L into pieces no larger than M which includes the integer 1, we can drop this 1 and (uniquely) generate a partition of $L - 1$ into pieces no larger than M . A generating function for $P_{L,M}$ is given analytically as

$$Z_M(q) = \prod_{m=1}^M \frac{1}{1 - q^m} = \sum_{L=1}^{\infty} q^L P_{L,M} \quad (\text{C2})$$

so that

$$P_{L,M} = \left[\frac{1}{L!} \left(\frac{d}{dq} \right)^L Z_M(q) \right]_{q=0} \quad (\text{C3})$$

APPENDIX D: PROOF OF LINEAR INDEPENDENCE

Linear independence of the $w_{\lambda,M}$ basis can be reformulated as follows: Given a polynomial P in the space spanned by the $w_{\lambda,M}$ (i.e, given a translationally invariant symmetric polynomial in M variables) this polynomial can be written as

$$P = \sum_{\lambda} a_{\lambda} w_{\lambda,M} \quad (D1)$$

where $\lambda_1 = 0$ in all terms of the sum (since the $w_{\lambda,M}$ basis spans the space). Proof of linear independence of all of the $w_{\lambda,M}$ basis vectors is equivalent to the statement that each such polynomial corresponds to a unique set of coefficients a_{λ} .

To prove this linear independence, we make use of translational invariance again and shift the center of mass coordinate of all M of the z_i 's to zero. Thus, we can rewrite Eq. 31 as

$$w_{\lambda,M} = \prod_{j=2}^M [e_{j,M}(z_1, \dots, z_{M-1}, \underline{z})]^{\lambda_j} \quad (D2)$$

where

$$\underline{z} = - \sum_{j=1}^{M-1} z_j \quad (D3)$$

Here, $w_{\lambda,M}$ must be a symmetric polynomial in the remaining $M-1$ variables z_j . Indeed, it is easy to check that

$$\begin{aligned} e_{m,M}(z_1, \dots, z_{M-1}, \underline{z}) &= e_{m,M-1}(z_1, \dots, z_{M-1}) \\ &- e_{m-1,M-1}(z_1, \dots, z_{M-1}) e_{1,M-1}(z_1, \dots, z_{M-1}) \\ &= e_{m,M-1} - e_{m-1,M-1} e_{1,M-1} \end{aligned} \quad (D4)$$

where again $e_{m,M-1}$ is defined to be zero for $m \geq M$. In particular note that $e_{M,M-1} = 0$ and

$$e_{M,M}(z_1, \dots, z_{M-1}, \underline{z}) = e_{M-1,M-1} e_{1,M-1} \quad (D5)$$

By rewriting our basis vectors $w_{\lambda,M}$ in terms of the $e_{m,M-1}$ we will be able to use the fact that the basis vectors generated by these $e_{m,M-1}$ (i.e., the $v_{\lambda,M-1}$ basis) are linearly independent.

We proceed as follows: Given the polynomial P written in terms of the w_{λ} basis vectors as in Eq. D1, we rewrite each w_{λ} as sums of products of $e_{m,M-1}$ using Eq. D4. I.e., we are rewriting everything in terms of the $v_{\lambda,M-1}$ basis. We then group terms such that $P = P_0 + P'_0$ where P_0 includes terms with no factors of $e_{1,M-1}$ and P'_0 includes terms with at least one factor of $e_{1,M-1}$. Writing P_0 in the $v_{\lambda,M-1}$ basis we have

$$P_0 = \sum_{\lambda^{(0)}} a_{\lambda^{(0)}}^{(0)} v_{\lambda^{(0)},M-1} \quad (D6)$$

where since P_0 has no factors of $e_{1,M-1}$, the vectors $\lambda^{(0)}$ must have $\lambda_1^{(0)} = 0$, and we must also have $\lambda_m^{(0)} = 0$ for $m \geq M$ since $e_{m,M-1} = 0$ for $m \geq M$. We then construct,

$$Q_0 = \sum_{\lambda^{(0)}} a_{\lambda^{(0)}}^{(0)} w_{\lambda^{(0)},M} \quad (D7)$$

with the basis $w_{\lambda,M}$ defined as in Eq. D2. Examining Eq. D4, we see that the $e_{m,M-1}$ terms generate precisely the polynomial P_0 within the expansion of Q_0 . The other terms in the expansion of Q_0 all contain $e_{1,M-1}$. Note that unlike P_0 , we have constructed Q_0 to be in the space spanned by $w_{\lambda,M}$ (i.e., it is a translationally invariant polynomial in M variables). Thus, since P is in the space spanned by $w_{\lambda,M}$ then $P - Q_0$ is also in the space spanned by $w_{\lambda,M}$. The difference $P - Q_0$ is constructed so as to have $e_{1,M-1}$ in all of its terms. Examining Eq. D4 and Eq. D5 we see that if a polynomial ($P - Q_0$ in this case) is in the space spanned by $w_{\lambda,M}$ and it has $e_{1,M-1}$ in all of its terms, it must also contain a factor of $e_{M-1,M-1}$ (i.e., this factor could only have come from a factor of $e_{M,M}$ in $w_{\lambda,M}$) or it must be zero. Thus, we can write

$$P = Q_0 - e_{1,M} e_{M-1,M-1} P_1 \quad (D8)$$

$$= Q_0 + e_{M,M}(z_1, \dots, z_{M-1}, \underline{z}) P_1 \quad (D9)$$

and now P_1 is a polynomial of degree less than that of P and is in the space spanned by $w_{\lambda,M}$. We then iterate this procedure, to rewrite P_1 as a sum of two terms, Q_1 containing no factors of $e_{1,M-1}$ and another term $e_{M,M} P_2$ with P_2 of lower degree still, and so forth. Eventually for some α_{max} , we must obtain $P_{\alpha_{max}} - Q_{\alpha_{max}} = 0$, which then terminates the procedure. Thus, we can successively decompose into a finite sum

$$P = \sum_{\alpha=0}^{\alpha_{max}} [e_{M,M}(z_1, \dots, z_{M-1}, \underline{z})]^{\alpha} Q_{\alpha} \quad (D10)$$

Where at each level of the iteration, we have

$$Q_{\alpha} = \sum_{\lambda^{(\alpha)}} a_{\lambda^{(\alpha)}}^{(\alpha)} w_{\lambda^{(\alpha)},M} \quad (D11)$$

where the vectors $\lambda^{(\alpha)}$ must have $\lambda_1^{(\alpha)} = \lambda_M^{(\alpha)} = 0$. We realize that Eqs. D10 and D11 can be recast in the form of Eq. D1 where the coefficients a_{λ} are determined by $a_{\lambda} = a_{\lambda^{(\alpha)}}^{(\alpha)}$ when $\lambda_M = \alpha$ and $\lambda_k^{(\alpha)} = \lambda_k$ for all $k \neq M$ (and we always have $\lambda_1^{(\alpha)} = \lambda_1 = 0$, and $\lambda_m^{(\alpha)} = \lambda_m = 0$ for $m > M$).

Our construction here is unique in the sense that given any polynomial P in the space spanned by $w_{\lambda,M}$ we find a unique set of coefficients a_{λ} in Eq. D1. This then completes our proof of the linear independence of the basis $w_{\lambda,M}$.

APPENDIX E: EXAMPLES: BOSONS WITH THREE-BODY INTERACTIONS

To elucidate our results, we will explicitly consider bosons interacting via three-body interactions. As discussed at length above, we will be concerned with three-body wavefunctions. In calculating any matrix element of these wavefunctions, we mean

$$\langle \Psi_1 | \hat{O} | \Psi_2 \rangle = \int d\mu [\Psi_1(z_1, z_2, z_3)]^* \hat{O} \Psi_2(z_1, z_2, z_3) \quad (\text{E1})$$

where \hat{O} is any operator and the integration measure is

$$d\mu = dz_1 dz_1^* dz_2 dz_2^* dz_3 dz_3^* \quad (\text{E2})$$

Recall that our full wavefunctions include both the translationally invariant relative wavefunction ψ as well as and the center of mass coordinate Ψ^{CM} , and the Gaussian factors (See Eq. 17)

$$\Psi_i = \psi_i^{CM} \left(\frac{z_1 + z_2 + z_3}{3} \right) \psi_i(z_1, z_2, z_3) e^{-\frac{1}{4} \sum_{i=1}^3 |z_i|^2} \quad (\text{E3})$$

Assuming that \hat{O} is translationally invariant, it only couples to ψ^{rel} . Thus, we can choose an orthogonal basis for ψ^{CM} and we will have the matrix element be zero if ψ_1^{CM} is orthogonal to ψ_2^{CM} . If we choose $\psi_1^{CM} = \psi_2^{CM}$ then, since the operator is translationally invariant, the matrix element should be independent of our different possible choice of ψ^{CM} , so long as the different possible choices are normalized the same way. For performing calculations, it is then acceptable to choose, for example, $\psi_1^{CM} = \psi_2^{CM} = 1$, and we can then check that our results are independent of this choice. This also implies a normalization choice for our matrix elements. We will work with this choice throughout this appendix.

For the relative wavefunctions, we will work in the $w_{\lambda,M}$ basis discussed above. We write out the first few three-cluster wavefunctions here explicitly¹⁸

$$\begin{aligned} |L=0\rangle &= w_{(0,0,0),3} = 1 \\ |L=2\rangle &= w_{(0,1,0),3} = e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \\ |L=3\rangle &= w_{(0,0,1),3} = e_{3,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \\ |L=4\rangle &= w_{(0,2,0),3} = [e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)]^2 \\ |L=5\rangle &= w_{(0,1,1),3} = e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) e_{3,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \\ |L=6, a\rangle &= w_{(0,3,0),3} = [e_{2,2}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)]^3 \\ |L=6, b\rangle &= w_{(0,0,2),3} = [e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)]^2 \\ |L=7\rangle &= w_{(0,2,1),3} = [e_{2,2}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)]^2 e_{3,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \\ &\vdots \end{aligned} \quad (\text{E4})$$

and recall the definition, Eq 30, of \tilde{z}_i in terms of z_i . Note that none of these basis states are normalized. Further, note that since $d_{sym}(L=6,3)=2$ there are two basis states for $L=6$. The two states, as discussed above, are linearly independent, but not orthogonal (or normalized).

We remind the reader that a wavefunction with relative angular momentum L vanishes as L powers when all three particles come to the same point.

We note in passing that for the particular case of three body wavefunctions, another basis has been constructed that is already orthonormalized¹⁹. However, this construction does not generalize easily to more than three particles, so we choose to work with the above basis to show the more general method.

We start by calculating the normalization of these basis states, by setting the operator $\hat{O} = 1$. The integrations over z_i 's can then be performed straightforwardly (Mathematica makes this trivial) to give the normalizations

$$\begin{aligned} \langle L=0 | L=0 \rangle &= 2^3 \pi^3 \\ \langle L=2 | L=2 \rangle &= 2^5 \pi^3 \\ \langle L=3 | L=3 \rangle &= 2^8 \pi^3 / 3^2 \\ \langle L=4 | L=4 \rangle &= 2^9 \pi^3 \\ \langle L=5 | L=5 \rangle &= 2^{12} \pi^3 / 3^2 \\ \langle L=7 | L=7 \rangle &= 5 * 2^{15} \pi^3 / 3^2 \end{aligned} \quad (\text{E5})$$

and for $L=6$ the full overlap matrix (See Eq. 22) is given by

$$\begin{pmatrix} \langle a|a \rangle & \langle a|b \rangle \\ \langle b|a \rangle & \langle b|b \rangle \end{pmatrix} = \pi^3 \begin{pmatrix} 3^2 * 2^{11} & -2^{12}/3 \\ -2^{12}/3 & 11 * 2^{13}/3^4 \end{pmatrix} \quad (\text{E6})$$

Example 1: As an example we now consider the toy model three-body point contact interaction

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \tilde{V} \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_2 - \mathbf{r}_3) \quad (\text{E7})$$

Note that this three-body interaction is fully reduced in the sense of Appendix B. Allowing the delta functions to act, the interaction matrix element then reduces to

$$\langle \Psi_1 | V | \Psi_2 \rangle = \tilde{V} \int dz dz^* [\psi_1(z, z, z)]^* \psi_2(z, z, z) e^{-3|z|^2/2} \quad (\text{E8})$$

From our basis states, only the state $|L=0\rangle = 1$ does not vanish when all three particles come to the same position, so the only nonzero matrix element is trivially calculated to be

$$\langle L=0 | V | L=0 \rangle = (2\pi/3) \tilde{V} \quad (\text{E9})$$

Thus, the only nonzero pseudopotential coefficient is $V_{0,3}$. In our non-normalized basis (See Eq. 24) we obtain

$$V_{0,3} = \frac{\langle L=0 | V | L=0 \rangle}{|\langle L=0 | L=0 \rangle|^2} = \frac{\tilde{V}}{3(2\pi)^5} \quad (\text{E10})$$

Alternately, we could construct a normalized basis (See Eq. 21)

$$|L=0\rangle_{Norm} = |L=0\rangle / \sqrt{8\pi^3} \quad (\text{E11})$$

in terms of which we would have (See 20) a pseudopotential coefficient

$$V_{0,3} =_{Norm} \langle L=0 | V | L=0 \rangle_{Norm} = \frac{\tilde{V}}{3(2\pi)^2} \quad (\text{E12})$$

As mentioned above, the highest density zero energy state of this interaction is precisely the Moore-Read Pfaffian state^{4,16}.

Example 2: Let us now consider a more complicated interaction. Consider

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \tilde{V} \nabla_1^4 [\delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_2 - \mathbf{r}_3)] \quad (\text{E13})$$

where the subscript 1 on ∇^4 means we are differentiating with respect to the position \mathbf{r}_1 . Derivative of delta function interactions, such as this one, are quite useful since arbitrary interactions can be built up as a series of successive derivatives of delta functions (We can think of such a series as a Taylor series expansion in fourier space).

To handle this interaction, we first integrate by parts then let one delta function act to give

$$\langle \Psi_1 | V | \Psi_2 \rangle = \tilde{V} \int d\mathbf{r}_1 d\mathbf{r}_2 \delta(\mathbf{r}_1 - \mathbf{r}_2) e^{-|\mathbf{r}_2|^2} \times \nabla_1^4 \left[[\psi_1(z_1, z_2, z_2)]^* \psi_2(z_1, z_2, z_2) e^{-|z_1|^2/2} \right] \quad (\text{E14})$$

Using the fact that

$$\nabla^2 = 4\partial_z \partial_{z^*} \quad (\text{E15})$$

the second line of Eq. E14 then becomes

$$(4^2) [(\partial_{z_1^*} - z_1/2)^2 [\psi_1(z_1, z_2, z_2)]^*] \times [(\partial_{z_1} - z_1^*/2)^2 \psi_2(z_1, z_2, z_2)] e^{-|z_1|^2/2} \quad (\text{E16})$$

Once the last delta function acts, all three of the particles are put at the same position. As we mentioned above, the wavefunction for a state with angular momentum L vanishes as L powers when all the particles come to the same position. Now, since we have up to two derivatives, we see that both $L = 0$ and $L = 2$ can have nonzero matrix elements, but no higher L wavefunction can. The $L = 0$ matrix element is messy to calculate (although it is easy on Mathematica) and gives

$$\langle L = 0 | V | L = 0 \rangle = (4^3 \pi / 27) \tilde{V} \quad (\text{E17})$$

The $L = 2$ matrix element, on the other hand, is actually easy to calculate. Here, we only need to keep the terms where all of the derivatives act on the wavefunctions (the $z/2$ and $z^*/2$ do not contribute). With our wavefunction being $e_{2,3}$, we note that

$$\begin{aligned} \partial_{z_1} e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) &= \partial_{\tilde{z}_1} e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = (\tilde{z}_2 + \tilde{z}_3) \\ \partial_{z_1}^2 e_{2,3}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) &= -2/3 \end{aligned} \quad (\text{E18})$$

so that

$$\partial_{z_1}^2 |L = 2\rangle = (-2/3) |L = 0\rangle \quad (\text{E19})$$

Thus the matrix element is given by $(-2/3)^2$ times the value of Eq. E9 times the prefactor of 4^2 from Eq. E15

$$\langle L = 2 | V | L = 2 \rangle = (4^2) (8\pi / 27) \tilde{V} \quad (\text{E20})$$

Again, we could write this in either the normalized, or unnormalized basis to give the pseudopotential coefficient $V_{2,3}$.

Example 3: Let us now try a much more complicated interaction. Consider

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \tilde{V} \nabla_1^{12} [\delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_2 - \mathbf{r}_3)] \quad (\text{E21})$$

The reason we choose this is because it has nontrivial matrix elements of $L = 6$, which is the first “interesting” case where there are two states at the same L . As above, we begin by integrating by parts. As in Example 2 above, calculation of matrix elements for $L < 6$ is quite messy, but for $L = 6$ things simplify quite a bit. Analogous to the $L = 2$ case for Example 2 above, the only term that does not vanish when the delta function acts is the one where all of the derivatives have been applied. Noting that from Eq. E4 we have $|L = 6, a\rangle = [e_{2,3}]^3$ and $|L = 6, b\rangle = [e_{3,3}]^2$ we then have

$$\alpha \equiv \partial_{z_1}^6 [e_{2,3}]^3 = \frac{6!}{2!2!2!} [\partial_{z_1}^2 e_{2,3}]^3 = -80/3$$

$$\beta \equiv \partial_{z_1}^6 [e_{3,3}]^2 = \frac{6!}{3!3!} [\partial_{z_1}^3 e_{3,3}]^2 = 320/81$$

Where we have used Eq. E18 above, as well as the analogous easily calculated $\partial_{z_1}^3 e_{3,3} = 4/9$. With these facts, to determine the matrix elements of V we realize that we have just the same integral as in Eq. E9 above times these factors (and a prefactor of 4^6 coming from Eq. E15)

$$\begin{pmatrix} \langle a | V | a \rangle & \langle a | V | b \rangle \\ \langle b | V | a \rangle & \langle b | V | b \rangle \end{pmatrix} = (4^6) (2\pi/3) \tilde{V} \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix} \quad (\text{E22})$$

Had we chosen to consider instead

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \tilde{V} \nabla_1^6 \nabla_2^6 [\delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_2 - \mathbf{r}_3)] \quad (\text{E23})$$

we would have obtained the same form of Eq. E22 but with

$$\alpha \equiv \partial_{z_1}^3 \partial_{z_2}^3 [e_{2,3}]^3 = 28/3 \quad (\text{E24})$$

$$\beta \equiv \partial_{z_1}^3 \partial_{z_2}^3 [e_{3,3}]^2 = 104/81 \quad (\text{E25})$$

Or analogously we might have chosen the interaction

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \tilde{V} \nabla_1^4 \nabla_2^4 \nabla_3^4 [\delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_2 - \mathbf{r}_3)] \quad (\text{E26})$$

to obtain the form of Eq. E22 but with

$$\alpha \equiv \partial_{z_1}^2 \partial_{z_2}^2 \partial_{z_3}^2 [e_{2,3}]^3 = -8/3 \quad (\text{E27})$$

$$\beta \equiv \partial_{z_1}^2 \partial_{z_2}^2 \partial_{z_3}^2 [e_{3,3}]^2 = 176/81 \quad (\text{E28})$$

and so forth (obviously there are many more possibilities in the same spirit).

It is interesting to note that independent of which of these interactions we choose, the form of Eq. E22 is rank one — meaning that it leaves some three-cluster wavefunctions with $L = 6$ at zero energy as discussed in section V. If we want to give positive energy to all three-cluster relative states with $L = 6$, we can do so by adding

together two of these interactions, for example adding the interaction in Eq. E21 to the interaction in Eq. E23 (and we should choose \tilde{V} positive in both cases).

Given the result in Eq. E22, and given the normalization matrix Eq. E6, we can plug into Eq. 24 to obtain the pseudopotential matrix $V_{6,3}^{r,r'}$ in this non-orthonormal basis. Or, we could construct an orthonormal basis using Eq. 21, and then construct the pseudopotential matrix $V_{6,3}$ in this basis as in Eq. 20.

Example 4: Let us now consider bosons in the first excited Landau level interacting with a three-body potential. Note that this is a bit of an artificial problem since most interesting boson problems would be lowest Landau level. Nonetheless, it is a well defined question of what the pseudopotentials would be for bosons in any given Landau level. Analogous to Eq. 11, we can raise any three-particle wavefunction into the first excited Landau level by applying $a_1^\dagger a_2^\dagger a_3^\dagger$ to the wavefunction, with the raising operator given by Eq. 12. To raise to the n^{th} excited Landau level, we would use $(a^\dagger)^n$.

We can thus raise the entire basis given in Eq. E4 to create a basis in the first excited Landau level, which we write as

$$\begin{aligned} |L=0\rangle_{1LL} &= a_1^\dagger a_2^\dagger a_3^\dagger |L=0\rangle = z_1^* z_2^* z_3^* / 2\sqrt{2} \\ |L=2\rangle_{1LL} &= a_1^\dagger a_2^\dagger a_3^\dagger |L=2\rangle = \dots \\ &\vdots \end{aligned} \quad (\text{E29})$$

with the unraised states on the right defined as in Eq. E4. We have written out the case of $L=0$ explicitly (with-

out the Gaussian factor), but the case of $L=2$ is a rather long expression. It is worth noting, however, that very generally, the raised function $|L\rangle_{1LL}$ does not vanish when all three particles come to the same position for $L \leq 3$ (the number 3 occurs here because there are 3 particles, therefore 3 raising operators, therefore a maximum of 3 derivatives).

It is convenient that the normalizations of these states are unchanged

$$\begin{aligned} {}_{1LL}\langle L=0|L=0\rangle_{1LL} &= \langle L=0|L=0\rangle \quad (\text{E30}) \\ {}_{1LL}\langle L=2|L=2\rangle_{1LL} &= \langle L=2|L=2\rangle \\ &\vdots \end{aligned}$$

since $[a^\dagger, a] = 1$, and LLL states plays the role of the a^\dagger vacuum.

Let us now consider the simple three point delta function interaction of Eq. E7. Allowing the delta functions to act we obtain the matrix elements as a single remaining integral as in Eq. E8. It is trivial to analytically obtain the result for $L=0$ and $L=3$ (we leave this as an exercise for the reader). For $L=2$, however, the calculation is done with the aid of Mathematica. We end up with

$$\begin{aligned} {}_{1LL}\langle L=0|V|L=0\rangle_{1LL} &= (4\pi/27)\tilde{V} \quad (\text{E31}) \\ {}_{1LL}\langle L=2|V|L=2\rangle_{1LL} &= (8\pi/9)\tilde{V} \\ {}_{1LL}\langle L=3|V|L=3\rangle_{1LL} &= (256\pi/243)\tilde{V} \end{aligned}$$

and matrix elements for higher L all vanish.

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¹⁴ To see more clearly that the matrix elements are diagonal in L , consider the behavior of the matrix element under the rotation generated by $z \rightarrow e^{i\theta} z$. If a rotationally invariant V is sandwiched between any two homogeneous polynomials of the same degree, the phase factors will cancel. However, if V is sandwiched between two polynomials of differing degree (differing L) the resulting matrix element will depend on $e^{i\theta}$ which implies either the matrix element is nonrotationally invariant, or it vanishes. Since we have assumed both V and the wavefunctions are rotationally invariant, the only possibility is that the matrix element between wavefunctions of differing L ’s vanishes.
¹⁵ See for example, I. G. MacDonald “Symmetric Functions and Hall Polynomials”, 2nd ed (Oxford University Press, NY) 1994; or
¹⁶ M. Greiter *et al.*, Nucl. Phys. B **374**, 567 (1992).

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¹⁸ Here we have a slight abuse of notation: the left hand side of Eq. E4 is a “ket” vector whereas the right hand side is

the explicit analytic part of the coordinate representation.
¹⁹ R. B. Laughlin, Phys. Rev. B. **27**, 3383 (1983).